Graphs of Functions

Symmetry

A graph is said to be **symmetric with respect to the x-axis** if, for every point \((x, y)\) on the graph, the point \((x, -y)\) is also on the graph.

If a graph is symmetric with respect to the x-axis and the point \((3, 5)\) is on the graph, then \((3, -5)\) is also on the graph.

Note: x-symmetry does not apply to functions of the form \(y = f(x)\) because we would fail the vertical line test.

Symmetry

A graph is said to be **symmetric with respect to the y-axis** if, for every point \((x, y)\) on the graph, the point \((-x, y)\) is also on the graph.

If a graph is symmetric with respect to the y-axis and the point \((3, 5)\) is on the graph, then \((-3, 5)\) is also on the graph.

A function symmetric with respect to the y-axis is referred to as an even function.
Symmetry

A graph is said to be **symmetric with respect to the origin** if, for every point \((x, y)\) on the graph, the point \((-x, -y)\) is also on the graph.

If a graph is symmetric with respect to the origin and the point \((3, 5)\) is on the graph then \((-3, -5)\) is also on the graph.

A function symmetric with respect to origin is referred to as an odd function.

Example

\[3x^2 - 2y = 25\]

To check for x-axis symmetry replace \(y\) with \(-y\):
\[3x^2 - 2(-y) = 25\]
\[3x^2 + 2y = 25\]

Since replacing \(y\) with \(-y\) produces a different equation, the equation is not symmetric about the x-axis.

Example

\[3x^2 - 2y = 25\]

To check for y-axis symmetry replace \(x\) with \(-x\):
\[3x^2 - 2y = 25\]
\[3(-x)^2 - 2y = 25\]
\[3x^2 - 2y = 25\]

Since replacing \(x\) with \(-x\) produces the same equation, the equation is symmetric about the y-axis.
Example

\[3x^2 - 2y = 25\]

To check for origin symmetry replace \(y\) with \(-y\) and \(x\) with \(-x\)

\[3x^2 - 2y = 25\]
\[3(-x)^2 - 2(-y) = 24\]
\[3x^2 + 2y = 24\]

Since replacing \(y\) with \(-y\) and \(x\) with \(-x\) produces a different equation, the equation is not symmetric about the origin.

Even Function

Rotation about the y-axis produces the same graph.

Odd Function

Rotation about the y-axis followed by rotation about the x-axis produces the same graph.
Neither

Rotation about the y-axis does not produce the same graph
Rotation about the y-axis followed by rotation about the x-axis does not produce the same graph

Example

\( y = 2x^4 - 4x^2 \)

Test for even/odd

\[
y(-x) = 2(-x)^4 - 4(-x)^2 \\
= 2x^4 - 4x^2 \\
= y(x)
\]

\( y(-x) = y(x) \Rightarrow \text{even} \)

Example

\( y = 2x^3 - 4x \)

Test for even/odd

\[
y(-x) = 2(-x)^3 - 4(-x) \\
= -2x^3 + 4x \\
= -(2x^3 - 4x) \\
= -y(x)
\]

\( y(-x) = -y(x) \Rightarrow \text{odd} \)
Example

\[ y = 2x^3 - 4x^2 \]

Test for even/odd

\[ y(-x) = 2(-x)^3 - 4(-x)^2 \]
\[ = -2x^3 - 4x^2 \]
\[ \neq -y(x) \text{ and } \neq y(x) \]

\[ y(-x) \neq y(x) \Rightarrow \text{ not even} \]
\[ y(-x) \neq -y(x) \Rightarrow \text{ not odd} \]

Hence neither

Match Symmetry

x-axis  y-axis  origin  neither

Positive and Negative Intervals

If \( f(x) > 0 \) for all \( a < x < b \) then \( f(x) \) is an positive on \( [a, b] \)

If \( f(x) < 0 \) for all \( a < x < b \) then \( f(x) \) is an negative on \( [a, b] \)
Find interval(s) where function is positive

Example

The graph is positive on interval (1, 4)

Find interval(s) where function is positive

Example

The graph is negative on interval (0, 1) and (4, 10)

Solving Absolute Value Inequalities

When the absolute value is greater than a number we use the conjunction OR

When the absolute value is less than a number we use the conjunction AND

Memory Aid - **GOR-LAND**

G (greater) OR

L (less than) AND

If > or ≥ the connecting word is “or”
If < or ≤ the connecting word is “and”
Absolute Value as a Distance

We can view the absolute value as a distance.

\[ |x - a| < d \] is valid for x-values that are less than distance d from a.

This is equivalent to \(-d < x - a < d\)

And to \(-d + a < x < d + a\)

Example

For \(|ax + b| < c\):

Solved using two columns

\[ ax + b < c \quad \text{and} \quad -(ax + b) < c \]
\[ ax < c - b \quad \text{and} \quad -ax - b < c \]
\[ x < \frac{c - b}{a} \quad \text{and} \quad -b - c < ax \]
\[ \frac{-b - c}{a} < x < \frac{c - b}{a} \]

Example

Solved using one columns

\[-c < ax + b < c \]
\[-c - b < ax < c - b \]
\[ \frac{-c - b}{a} < x < \frac{c - b}{a} \]

in interval notation \(\left(\frac{-c - b}{a}, \frac{c - b}{a}\right)\)
Example

Solved as a distance \[ |a| \left| x - \frac{-b}{a} \right| < c \]
\[ \left| x - \frac{-b}{a} \right| < \left| \frac{c}{a} \right| \]

As a distance this is valid when \( x \) is less than \( \frac{c}{|a|} \) from \( -\frac{b}{a} \)

Then
\[ -\frac{c}{|a|} - \frac{b}{a} < x < \frac{c}{|a|} - \frac{b}{a} \]

Example

\[ |3x + 2| \leq 8 \]

Solved using one columns
\[ -8 \leq 3x + 2 \leq 8 \]
\[ -10 \leq 3x \leq 6 \]
\[ -\frac{10}{3} \leq x \leq 2 \]

Solved as a distance
\[ \left| x - \left( \frac{-2}{3} \right) \right| \leq \frac{8}{3} \]
\[ -\frac{8}{3} \leq x \leq \frac{8}{3} \]
\[ -\frac{8}{3} \leq x \leq \frac{8}{3} \]

Then
\[ -\frac{10}{3}, 2 \]

Example

\[ |3x + 2| \leq 8 \]

Solved using the TI-84

\[ x = -\frac{10}{3} \]
\[ x = 2 \]
Example

We graph $|x - 3| - |x + 4|$

We want to find $x$-values $|x - 3| - |x + 4| > 0$
Example

\[ |x - 3| > |x + 4| \]

Our solution \( \{ x \mid x > 3 \frac{1}{2} \} \)
Remarks

On a given interval, if the graph of a function rises from left to right, it is said to be increasing on that interval.

If the graph drops from left to right, it is said to be decreasing.

If the function values stay the same from left to right, the function is said to be constant.

Increasing Interval

A function with a graph that goes up as it is followed from left to right. For example, any line with a positive slope is increasing.

A function \( f \) is increasing on the interval \( I \) if, for each \( a < b \) in \( I \), \( f(a) < f(b) \).

Example

Find interval increasing

increasing on \((3, \infty)\)
Decreasing Interval
A function with a graph that moves downward as it is followed from left to right. For example, any line with a negative slope is decreasing.

A function $f$ is decreasing on the interval $I$ if, for each $a < b$ in $I$, $f(a) > f(b)$.

Example
Find interval decreasing on $(1, \infty)$

Constant Interval
A function with a graph that moves horizontally as it is followed from left to right. For example, any constant line with a zero slope is constant.

A function $f$ is constant on the interval $I$ if, for each $a < b$ in $I$, $f(a) = f(b)$. 
Classical of Functions

If \( f(a) = f(b) \) for all \( a, b \) then \( f(x) \) is a constant function

If \( f(a) > f(b) \) for all \( a > b \) then \( f(x) \) is an increasing function

If \( f(a) < f(b) \) for all \( a > b \) then \( f(x) \) is a decreasing function

If \( f(a) \geq f(b) \) for all \( a > b \) then \( f(x) \) is a non-decreasing function

If \( f(a) \leq f(b) \) for all \( a > b \) then \( f(x) \) is a non-increasing function

Example

Find intervals increasing and decreasing

Increasing on \( (-\infty, 3) \)
Decreasing on \( (3, \infty) \)

Example

Find increasing, decreasing and constant intervals

The graph is increasing on an interval \( (0, 2) \)
The graph is decreasing on an interval \( (2, 7) \)
The graph is constant on an interval \( (7, 10) \)
For the function $f$ defined by $f(x) = -3x^2 + 2x$, evaluate:

$$\frac{f(x+h) - f(x)}{h} = \frac{(-3(x+h)^2 + 2(x+h)) - (-3x^2 + 2x)}{h}$$

$$= \frac{(-3x^2 + 6xh - 3h^2 + 2x + 2h) - (-3x^2 + 2x)}{h}$$

$$= \frac{-6xh - 3h^2 + 2h}{h} = -6x - 3h + 2$$
**Difference Quotient**

\[ f(x + h) - f(x) \]

\[ \frac{f(x + h) - f(x)}{h} \]

This is the average rate of change (ARC) through the points \((x, f(x))\) and \((x+h, f(x+h))\).

The ARC is the slope of this line.

It is an extremely important concept in Calculus.

**Example**

Find the difference quotient of \(y = \sqrt{2x+1}\) at \((4, 3)\).

\[ f(x) = f(4) = 3 \]

\[ f(x + h) = f(4 + h) = \sqrt{2(4 + h) + 1} = \sqrt{9 + 2h} \]

\[ f(x + h) - f(x) = \sqrt{9 + 2h} - 3 \]

\[ \frac{f(x + h) - f(x)}{h} = \frac{\sqrt{9 + 2h} - 3}{h} \]
Example

Find the difference quotient of \( y = \sqrt{2x + 1} \) from \( x = 4 \) to \( x = 4.1 \)

\[
\frac{f(x + h) - f(x)}{h} = \frac{\sqrt{9+2h} - 3}{h} \quad \text{with} \quad h = 0.1
\]

\[
= \frac{\sqrt{9+2(0.1)} - 3}{0.1} = \frac{3.033 - 3}{0.1} = \frac{0.033}{0.1} = 0.33
\]

Practice

Write the derivative of \( f(x) = 3x^2 - 2x + 1 \)