Properties of Functions

Learning Objectives
1. Determine even and odd functions from a graph
2. Identify even and odd functions from the equation
3. Use a graph to determine where a function is increasing, decreasing, or constant
4. Use a graph to locate local maxima and local minima
5. Use a graph to locate the absolute maximum and the absolute minimum
6. Use a graphing utility to approximate local maxima and local minima and to determine where a function is increasing or decreasing
7. Find the average rate of change of a function

Review
A graph is said to be even if for every point \((x, y)\) on the graph, the point \((-x, y)\) is also on the graph.

If a graph is even and the point \((3, 5)\) is on the graph then \((-3, 5)\) is also on the graph.
A graph is said to be odd if, for every point \((x, y)\) on the graph, the point \((-x, -y)\) is also on the graph.

If a graph is odd and the point \((3, 5)\) is on the graph then \((-3, -5)\) is also on the graph.

\[
y = 2x^4 - 4x^2
\]

Test for even/odd
\[
y(-x) = 2(-x)^4 - 4(-x)^2
\]
\[
= 2x^4 - 4x^2
\]
\[
= y(x)
\]

\[y(-x) = y(x) \Rightarrow \text{even}\]

\[
y = 2x^3 - 4x
\]

Test for even/odd
\[
y(-x) = 2(-x)^3 - 4(-x)
\]
\[
= -2x^3 + 4x
\]
\[
= -(2x^3 - 4x)
\]
\[
= -y(x)
\]

\[y(-x) = -y(x) \Rightarrow \text{odd}\]
Example

\[ y = 2x^3 - 4x^2 \]

Test for even/odd

\[
y(-x) = 2(-x)^3 - 4(-x)^2
\]

\[= -2x^3 - 4x^2\]

\[\neq -y(x) \text{ and } \neq y(x)\]

\[y(-x) \neq y(x) \Rightarrow \text{not even}\]

\[y(-x) \neq -y(x) \Rightarrow \text{not odd}\]

\[\Rightarrow \text{hence neither}\]

Even Function

Rotation about the y-axis produces the same graph

Odd Function

Rotation about the y-axis followed by rotation about the x-axis produces the same graph
Neither

Rotation about the y-axis does not produce the same graph
Rotation about the y-axis followed by rotation about the x-axis does not produce the same graph

Examples

\(-2 < x \leq 0\)

Our graph

\[\begin{array}{c}
\begin{array}{ccccccc}
& & & & & & \\
| & & & & & & \\
-5 & -4 & -3 & -2 & 0 & 1 & 2 \\
\end{array}
\end{array}\]

Set Builder Notation \( \{x \mid -2 < x \leq 0\} \)
Interval Notation \((-2, 0]\)

Examples

\(x < 3\)

Our graph

\[\begin{array}{c}
\begin{array}{ccccccc}
& & & & & & \\
| & & & & & & \\
-5 & -4 & -3 & -2 & 0 & 1 & 2 \\
\end{array}
\end{array}\]

Set Builder Notation \( \{x \mid x < 3\} \)
Interval Notation \((\infty, 3)\)
Example

-1.5 < x < 3

Our graph

Set Builder Notation \( \{ x \mid -1.5 < x < 3 \} \)

Interval Notation \((-1.5, 3)\)

Positive and Negative Intervals

If \( f(x) > 0 \) for all \( a < x < b \) then \( f(x) \) is positive on \([a, b]\)

If \( f(x) < 0 \) for all \( a < x < b \) then \( f(x) \) is negative on \([a, b]\)

Example

Find interval(s) where function is positive

The graph is positive on interval (1,4)
Remarks

On a given interval, if the graph of a function rises from left to right, it is said to be increasing on that interval.

If the graph drops from left to right, it is said to be decreasing.

If the function values stay the same from left to right, the function is said to be constant.

Increasing Interval

A function with a graph that goes up as it is followed from left to right. For example, any line with a positive slope is increasing.

A function $f$ is increasing on the interval $I$ if, for each $a < b$ in $I$, $f(a) < f(b)$.

Example

Find interval increasing

increasing on $(3, \infty)$
Decreasing Interval

A function with a graph that moves downward as it is followed from left to right. For example, any line with a negative slope is decreasing.

A function \( f \) is decreasing on the interval \( I \) if, for each \( a < b \) in \( I \), \( f(a) > f(b) \).

Example

Find interval decreasing

Constant Interval

A function with a graph that moves horizontally as it is followed from left to right. For example, any constant line with a zero slope is constant.

A function \( f \) is constant on the interval \( I \) if, for each \( a < b \) in \( I \), \( f(a) = f(b) \).
Classification of Functions

If \( f(a) = f(b) \) for all \( a, b \) then \( f(x) \) is a constant function

If \( f(a) > f(b) \) for all \( a > b \) then \( f(x) \) is an increasing function

If \( f(a) < f(b) \) for all \( a > b \) then \( f(x) \) is a decreasing function

If \( f(a) \geq f(b) \) for all \( a > b \) then \( f(x) \) is a non-decreasing function

If \( f(a) \leq f(b) \) for all \( a > b \) then \( f(x) \) is a non-increasing function

Example

Find intervals increasing and decreasing

Increasing on \((-\infty, 3)\)
Decreasing on \((3, \infty)\)

Example

Find increasing, decreasing and constant intervals

The graph is increasing on an interval \((0, 2)\)
The graph is decreasing on an interval \((2, 7)\)
The graph is constant on an interval \((7, 10)\)
Example

Find interval(s) where function is positive

The graph is negative on interval (0,1) and (4,10)

Maxima and Minima

In mathematics, maxima and minima, known collectively as extremum, are the largest value (maximum) or smallest value (minimum), that a function takes in a point either within a given neighborhood (local extremum) or on the function domain in its entirety (global extremum).

local maxima will look like the tops of hills

local minima will look like the bottoms of valleys
Example

Find extrema

The graph has maximum \( y = 3 \)
The graph has minimum \( y = -3 \)

Example

Find min, max, where increasing/decreasing \( f(x) = x^3 - 3x + 3 \) on \((-2, 2)\)

Enter function into graphing calculator

Set window for given interval

Example

Find min, max, where increasing/decreasing \( f(x) = x^3 - 3x + 3 \) on \((-2, 2)\)

Observe graph

We want \((x, y)\) for four points
Example Find min, max, where increasing/decreasing $x^3 - 3x + 3$ on $(-2, 2)$

End points come from the table

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

Example Find min, max, where increasing/decreasing $x^3 - 3x + 3$ on $(-2, 2)$

Compute relative maximum

Relative maximum is $(-1, 5)$

Example Find min, max, where increasing/decreasing $x^3 - 3x + 3$ on $(-2, 2)$

Compute relative minimum

Relative minimum is $(1, 1)$
Example: Find min, max, where increasing/decreasing $x^3 - 3x + 3$ on $(-2, 2)$

Min is $y=1$ at $x=-2$ and $x=1$
Max is $y=5$ at $x=-1$ and $x=2$
Decreasing $(-1, 1)$
Increasing $(-2, -1)$ and $(1, 2)$

Example: Solve system of equations

\[
\begin{align*}
\begin{cases}
x + y &= 5 \\
x - y &= 1
\end{cases}
\end{align*}
\]

Rewrite as matrix

\[
\begin{bmatrix}
1 & 1 & 5 \\
1 & -1 & 1
\end{bmatrix}
\]

Enter Matrix

Example: Solve system of equations

\[
\begin{align*}
\begin{cases}
x + y &= 5 \\
x - y &= 1
\end{cases}
\end{align*}
\]

From home screen

Solution $(3, 2)$
Example

Solve system of equations
\[ \begin{align*}
  x + y &= 5 \\
  x - y &= 1
\end{align*} \]

If not exact, \[ \text{[MATH]} \text{[ENTER]} \text{[ENTER]} \] for fraction

Secant Line

A secant line, also simply called a secant, is a line passing through two points of a curve

The word secant comes from the Latin secare

Example

Find equation of secant line through \((-3, 25)\) and \((2,10)\)

\[
\begin{align*}
  m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 25}{2 - (-3)} = \frac{-15}{5} = -3 \\
  y - y_1 &= m(x - x_1) \\
  y - 10 &= -3(x - 2) = -3x + 6 \\
  y &= -3x - 4
\end{align*}
\]
**Average Rate of Change**

The average rate of change (ARC) is the slope of the line through two points on a curve.

\[ \text{ARC} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]

**Remarks**

In Calculus we will refer to the line through two points of a function as the secant line.

The dashed line is the secant line. The slope of this line is the Average Rate of Change (ARC).

**Example**

Find equation of secant line through \((-3, 25)\) and \((2, 10)\)

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 25}{2 - (-3)} = \frac{-15}{5} = -3 \]

\[ y - y_1 = m(x - x_1) \]

\[ y - 25 = -3(x + 3) \]

\[ y = -3x - 13 \]
Average Rate of Change

Suppose we want to find the ARC using the points \((x, f(x))\) and \((x+h, f(x+h))\)

Then \(x_1 = x\), \(y_1 = f(x)\) and \(x_2 = x+h\), \(y_2 = f(x+h)\)

We then calculate the ARC using the equation for \(m\)

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}
\]

\[
ARC = \frac{f(x+h) - f(x)}{h}
\]

---

Average Rate of Change

The Average Rate of Change (ARC) is the change in the value of a quantity divided by the elapsed time.

For a function, this is the change in the \(y\)-value divided by the change in the \(x\)-value for two distinct points on the graph.

This is the same thing as the slope of the secant line that passes through the two points.

For a linear function the ARC is the slope.

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Example

If you travel about 75 miles in one and a half hours what is your average speed?

\[
\text{average speed} = \frac{75 \text{ miles}}{1.5 \text{ hours}} = 50 \text{ mph}
\]
Example
Find average rate of change of \( y = x^2 + 1 \) on \([2, 5]\)

\[
y(2) = (2)^2 + 1 = 4 + 1 = 5 \]
\[
y(5) = (5)^2 + 1 = 25 + 1 = 26 \]
\[
\frac{y(5) - y(2)}{5 - 2} = \frac{26 - 5}{3} = \frac{21}{3} = 7
\]

Example
Find average rate of change of \( y = x^2 + 1 \) on \([2, 5]\)

The average rate of change is slope of the solid red line
Example

Find equation of secant line of \( y = x^2 + 1 \) through \([2, 5]\)

\[ y - y_1 = m(x - x_1) \]
\[ y - 5 = 7(x - 2) \]
\[ y = 7x - 9 \]

Difference Quotient

The ARC is calculated using the points and the difference quotient

\[ ARC = \frac{f(x + h) - f(x)}{h} \]

This is an important equation in Math 2413

Finding the Difference Quotient

Step 1 – Find \( f(x + h) \)
Step 2 – Find \( f(x) \)
Step 3 – Find \( f(x + h) - f(x) \)
Step 4 – Divide by \( h \)

Simplify at each step along the way
Example
Find the Difference Quotient of $f(x) = 2x^2 + 4$

Step 1 – Find $f(x+h)$
$$f(x + h) = 2(x+h)^2 + 4$$
$$= 2(x^2 + 2xh + h^2) + 4$$
$$= 2x^2 + 4xh + 2h^2 + 4$$

Step 2 – Find $f(x)$
$$f(x) = 2x^2 + 4$$

Step 3 – Find $f(x+h) - f(x)$
$$f(x + h) - f(x) = (2x^2 + 4xh + 2h^2 + 4) - (2x^2 + 4)$$
$$= 4xh + 2h^2$$

Step 4 – Divide by $h$
$$\frac{f(x + h) - f(x)}{h} = \frac{4xh + 2h^2}{h} = 4x + 2h$$