Graphing Techniques and Transformations

Learning Objectives

1. Graph functions using vertical and horizontal shifts
2. Graph functions using compressions and stretches
3. Graph functions using reflections about the $x$-axis and the $y$-axis

Remarks

We can use our basic functions to create new functions by

- Horizontal Translations
- Vertical Translations
- Reflection across $x$-axis
- Reflection across $y$-axis
- Stretching/Shrinking along $x$-coordinate
- Stretching/Shrinking along $y$-coordinate
Translation: \( f(x) + k \)

Graph \( y_1 = x^2 \) on your graphing calculator.

We will then compare this with different graphs to generalize the effect of \( k \).

- \( k = -4 \)
- \( k = -2 \)
- \( k = 2 \)
- \( k = 4 \)
Translation: \( f(x) + k \)

Graph \( y_1 = x^2 \) on your graphing calculator and then graph \( y_2 \) given below to determine the movement of the graph of \( y_2 \) as compared to \( y_1 \). Generalize the effect of \( k \)

<table>
<thead>
<tr>
<th>( y_2 )</th>
<th>Direction of Translation</th>
<th>Units Translated</th>
<th>Value of ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - 4 )</td>
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<tr>
<td>( x^2 - 2 )</td>
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<tr>
<td>( x^2 + 2 )</td>
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Vertical Translations

\[ y = f(x - h) + k \]
\[ y - k = f(x - h) \]

- The value of \( k \) causes the graph of \( f(x) \) to translate up or down (vertically)
- If \( k > 0 \), the graph shifts \( k \) units up
- If \( k < 0 \) then the graph shifts \( k \) units down

Translation: \( f(x - h) \)

Graph \( y_1 = x^2 \) on your graphing calculator

We will then compare this with different graphs to generalize the effect of \( h \)
Translation: $f(x - h)$

$h = 4$

$h = 2$

$h = -2$

$h = -4$

Graph $y_1 = x^2$ on your graphing calculator and then graph $y_2$ given below to determine the movement of the graph of $y_2$ as compared to $y_1$. Generalize the effect of $h$.

<table>
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<tr>
<td>$(x - 4)^2$</td>
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<tr>
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Horizontal Translations

\[ y = f(x-h) + k \]
\[ y - k = f(x-h) \]

- The value of \( h \) causes the graph of \( f(x) \) to translate left or right (horizontally)
- If \( h < 0 \), the graph shifts \( h \) units left
- If \( h > 0 \), the graph shifts \( h \) units right

Horizontal and Vertical Translations

\[ f(x-h) + k \]

- The value of \( h \) causes the graph of \( f(x) \) to translate left or right (horizontally)
  - If \( h < 0 \), the graph shifts \( h \) units left
  - If \( h > 0 \), the graph shifts \( h \) units right

- The value of \( k \) causes the graph of \( f(x) \) to translate up or down (vertically)
  - If \( k > 0 \), the graph shifts \( k \) units up
  - If \( k < 0 \) then the graph shifts \( k \) units down

x- and y-Axis Reflections

- The graph of \( y = -f(x) \) is the same as graph of \( f(x) \) but reflected about the \( x \)-axis
- The graph of \( y = f(-x) \) is the same as graph of \( f(x) \) but reflected about the \( y \)-axis
Caution in Translations of Graphs

- The order in which transformations are made is important.
- If they are made in a different order, a different equation can result.
  - For example, the graph of $y = 2|x + 3|$ is obtained by \textbf{first} stretching the graph of $y = |x|$ by a factor of 2, and \textbf{then} translating 3 units upward.
  - The graph of $y = 2|x + 3|$ is obtained by \textbf{first} translating horizontally 3 units to the left, and \textbf{then} stretching by a factor of 2.

Vertical Compression: $y = a \cdot f(x)$

Graph $y_1 = x^2$ on your graphing.

We will generalize the effect of $a$.

Vertical Compression: $y = a \cdot f(x)$

- $a = 1/2$
- $a = 1/3$
**Vertical Compression:  \( y = a \cdot f(x) \)**

Graph \( y_1 = x^2 \) on your graphing calculator and then graph \( y_2 \) given below to determine the shape of the graph of \( y_2 \) as compared to \( y_1 \). Generalize the effect of \( a \)

<table>
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<td>( (1/2)x^2 )</td>
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**Vertical Stretch:  \( y = a \cdot f(x) \)**

Graph \( y_1 = x^2 \) on your graphing calculator

We will generalize the effect of \( a \)

\[ a = 2 \]

\[ a = 3 \]
Graph $y_1 = x^2$ on your graphing calculator and then graph $y_2$ given below to determine the shape of the graph of $y_2$ as compared to $y_1$. Generalize the effect of $a$

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<td>$2x^2$</td>
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<td></td>
</tr>
<tr>
<td>$3x^2$</td>
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**Vertical Stretch: $y = a \cdot f(x)$**

The graph of $y = a \cdot f(x)$ is obtained from the graph of $y = f(x)$ by

- Vertically stretching the graph if $|a| > 1$
- Vertically compressing the graph if $0 < |a| < 1$

As $a$ get larger positive it appear narrower
As $a$ get smaller positive it appear wider

As $a$ get larger negative it appear narrower
As $a$ get smaller negative it appear wider
Horizontal Stretches: $y = f(a \cdot x)$

Graph $y_1 = x^2$ on your graphing.

We will generalize the effect of $a$

- $a = 1/2$

- $a = 1/3$

- $a = 2$

- $a = 3$
**Horizontal Stretches: \( y = f(a \cdot x) \)**

Graph \( y_1 = x^2 \) on your graphing. Generalize the effect of \( a \)

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**Horizontal Stretches: \( y = f(a \cdot x) \)**

The graph of \( y = f(a \cdot x) \) is obtained from the graph of \( y = f(x) \) by
- horizontally compressing the graph if \(|a| > 1\)
- horizontally stretching the graph if \(0 < |a| < 1\)

**Reflection Across x-axis: \( y = -f(x) \)**

Graph \( y_1 = \sqrt{x} \) on your graphing

We will learn the effect of reflection across the x-axis
Reflection Across y-axis: \( y = f(-x) \)

Graph \( y = \sqrt{x} \) on your graphing

We will learn the effect of reflection across the y-axis

x- and y-Axis Reflections

- The graph of \( y = -f(x) \) is the same as graph of \( f(x) \) but reflected about the x-axis.

- The graph of \( y = f(-x) \) is the same as graph of \( f(x) \) but reflected about the y-axis.

General Transformation

Given \( y = f(x) \) the graph of \( y = af(x-h)+k \)

Can be obtained by the following sequence of transformations

- Horizontal shift (to right if \( h > 0 \))
- Reflection across x-axis if \( a < 0 \)
- Stretch if \( |a| > 1 \)
- Compression (shrink) if \( |a| < 1 \)
- Vertical shift (up if \( k > 0 \))
Sequence of Transformations

Point – by – Point Method

• Follow order of operations
• Select two points (or more) from the original function and move that point one step at a time
• Plot these points and sketch the new graph

Example

Given \( f(x) = x^3 \) find

\[ 3f(x+2) - 1 = 3(x+2)^3 - 1 \]

\( f(x) \) contains (-1,-1), (0,0), (1,1)

1\(^{st}\) transformation would be \((x+2)\), which moves the function left 2 units (subtract 2 from each x), pts. are now (-3,-1), (-2,0), (-1,1)

2\(^{nd}\) transformation would be 3 times all the y's, pts. are now (-3,-1), (-2,0), (-1,3)

3\(^{rd}\) transformation would be subtract 1 from all y's, pts. are now (-3,-2), (-2,-1), (-1,2)

Example

Given \( f(x) = x^3 \) find

\[ 3f(x+2) - 1 = 3(x+2)^3 - 1 \]

Begin with \( f(x) = x^3 \)
Create \[ 3f(x+2) - 1 = 3(x+2)^3 - 1 \]
Example

Given \( f(x) = x^3 \) find
\[ 3f(x + 2) - 1 = 3(x + 2)^3 - 1 \]

\( f(x) = x^3 \)

Translate left 2 units

\( f(x) = (x + 2)^3 \)

Example

Given \( f(x) = x^3 \) find
\[ 3f(x + 2) - 1 = 3(x + 2)^3 - 1 \]

\( f(x) = (x + 2)^3 \)

Stretch x 3 vertically

Example

Given \( f(x) = x^3 \) find
\[ 3f(x + 2) - 1 = 3(x + 2)^3 - 1 \]

\( f(x) = 3(x + 2)^3 \)

Translate 1 unit down

\( f(x) = 3(x + 2)^3 - 1 \)
Example

Given \( f(x) = x^3 \) find

\[ 3f(x+2) - 1 = 3(x+2)^3 - 1 \]

We can select several points from \( f(x) = x^3 \)

\((-1,-1), (0,0), (1,1), (2,8)\) and using \( f(x) = 3(x+2)^3 - 1 \)

We have:

| \(-1, -1\) | \((-1+2)^3 - 1 = 2\) |
| \((0,0)\)   | \((0+2)^3 - 1 = 23\) |
| \((1,1)\)   | \((1+2)^3 - 1 = 80\) |
| \((2,8)\)   | \((2+2)^3 - 1 = 191\) |

Example

Given \( f(x) = x^3 \) find

\[ 3f(x+2) - 1 = 3(x+2)^3 - 1 \]

Example

Given \( f(x) = x^3 \) find

\[ 3f(x+2) - 1 = 3(x+2)^3 - 1 \]