The Graph of a Rational Function

Learning Objectives

1. Analyze the graph of a rational function
2. Solve applied problems involving rational functions

Remarks

Vertical asymptotes occur at any x-values that make the denominator 0.

The graph of a rational function never crosses a vertical asymptote.
y-intercept

If \( R(x) = \frac{p(x)}{q(x)} = \frac{p(x)}{q(x)} \) after factoring, then the

\[
\text{If } 0 \in D(R(x)) \text{ there is not y-int. Otherwise}
\]

the y-int is \( y = R(0) = \frac{p(0)}{q(0)}, q(0) \neq 0 \)

or \( y = R(0) = \frac{p(0)}{q(0)}, q(0) \neq 0 \)

Example

Find y-intercept \( f(x) = \frac{(x+1)(x-1)}{(x+1)(x+4)} \)

\[
f(0) = \frac{(0+1)(0-1)}{(0+1)(0+4)} = \frac{-1}{4}
\]

x-intercepts

If \( R(x) = \frac{p(x)}{q(x)} = \frac{p(x)}{q(x)} \) after factoring, then

the zeroes of \( p(x) = 0 \) are x-intercepts

We do not find zeros of \( p(x) \) because some of these zeros are not in the domain, and hence are not x-intercepts
Example
Find x-intercept(s) \( f(x) = \frac{(x+1)(x-1)}{(x+1)(x+4)} \)

\[
= \frac{x-1}{x+4}
\]

x-intercept is \( x = 1 \)

Nomenclature
\( x \to c^- \) mean \( x \) approaches \( c \) from the left and
\( x \to c^+ \) means \( x \) approaches \( c \) from the right

So \( x \to 0^- \Rightarrow f(x) \to -\infty \)

means as \( x \) approaches 0 from the left that
\( f(x) \) approaches negative infinity

Example
\( f(x) = \frac{1}{x} \)

\( x \to 0^- \Rightarrow f(x) \to -\infty \)

\( x \to 0^+ \Rightarrow f(x) \to \infty \)
Remarks

If \( x \to a^- \Rightarrow f(x) \to c \)
and
If \( x \to a^+ \Rightarrow f(x) \to c \)

Then we can write \( x \to a \Rightarrow f(x) \to c \)

In calculus, we refer to this process, as taking the limit.

Review

\( x \to a^- \) means \( x \) approaches \( a \) from the left
\( x \to a^+ \) means \( x \) approaches \( a \) from the right

Example

State behavior as \( x \) approaches 4

\( x \to 4^- \Rightarrow f(x) \to -\infty \)
\( x \to 4^+ \Rightarrow f(x) \to \infty \)
Graphing Rational Functions

**STEP 1 - Domain**
Find Domain, solve \( q(x) = 0 \)

**STEP 2 – Vertical Asymptotes and Holes**
Reduce \( R(x) = \frac{p(x)}{q(x)} \) to \( p'(x) = \frac{q'(x)}{q(x)} \) and find holes
Find vertical asymptotes, solve \( q'(x) = 0 \)

**STEP 3 – Horizontal Asymptote**
If \( n = d \), then \( y = \frac{a_n}{b_n} \) is horizontal asymptote
If \( n < d \), then \( y = 0 \) is horizontal asymptote

**STEP 4 – Oblique (Slant) Asymptote**
If \( n = d + 1 \), divide \( p'(x) \) by \( q'(x) \) to find \( R(x) = f(x) + \frac{r(x)}{q'(x)} \)
then \( y = f(x) \) is the oblique asymptote

**STEP 5 – Intercepts**
Solve \( p'(x) = 0 \) to find x-intercepts
To find y-intercept set \( y = R(0) \)

**Example**
Analyze \( f(x) = \frac{x^2 - 6x + 12}{x - 4} \)

\( x - 4 \neq 0 \Rightarrow \text{Domain} = \mathbb{R} \setminus \{4\} \)
Vertical asymptote at \( x = 4 \)
\( y = \text{int} = f(0) = \frac{0^2 - 6(0) + 12}{0 - 4} = -3 \)
\( x^2 - 6x + 12 = 0 \) no real solutions \( \Rightarrow \) no x-int
\( x - 4 \bigg| x^2 - 6x + 12 \Rightarrow y = x - 2 \) oblique asymptote

\[
\begin{align*}
\frac{x}{x^2 - 4x} & = \frac{x}{-2x + 12} \\
& = \frac{x}{-2x + 8} \\
& = \frac{x}{4}
\end{align*}
\]
Example \[\frac{x^3-3x^2-4x}{x^2+3x}\]

\[
x^3-3x^2-4x = \frac{x(x-4)(x+1)}{x(x+3)} = \frac{(x-4)(x+1)}{x+3}
\]

Domain \((x+3)=0 \Rightarrow \) Domain all reals except \(x=-3,0\)

\(x+3=0 \Rightarrow x=-3\) is vertical asymptotes

and \(x=0\) is holes

\(n>d \Rightarrow \) no horizontal asymptote

\[
\frac{x^3-3x^2-4x}{x^2+3x} = x - 6 + \frac{14}{x+3}
\]

\(\Rightarrow y = x - 6\) is oblique asymptote

\((-\infty, 2)\) increasing

\([2]\) relative maximum

\((2,4)\) decreasing

\([4]\) undefined

\((4,5)\) decreasing

\([6]\) relative minimum

\((6,\infty)\) increasing

\((x-4)(x+1)=0 \Rightarrow x=-1,4\) are x-intercepts

note \(x=0\) is hole, so no x-intercept possible

we should recognize it is not in the domain

\(f(0)\) is undefined as \(x=0\) (a hole) is not in the domain, hence there is no y-intercept
Example

\[ \frac{x-1}{x^2-4} = \frac{x-1}{(x+2)(x-2)} \]

\[(x+2)(x-2) = 0 \Rightarrow D(f) = \mathbb{R} - \{-2, 2\}\]

Since \(x+2\) and \(x-2\) do not factor out

\(x = -2\) and \(x = 2\) are VA and there are no holes

\(n < d \Rightarrow y = 0\) is HA

\(x-1=0 \Rightarrow x=1\) is an x-int

\(0-1 = y = \frac{1}{4}\) is the y-int

Example

\[ \frac{x-1}{x^2-4} \]

We see the VA

We see the where D(f) not defined

We see the VA is \(y = 0\)

Example

\[ \frac{3x^2-3x}{x^2+x-12} = \frac{3x(x-1)}{(x+4)(x-3)} \]

\((x+4)(x-3) = 0 \Rightarrow D(f) = \mathbb{R} - \{-4, 3\}\)

Neither factor out, so \(x = -3, x = 4\) are VA

\(n=d \Rightarrow y = \frac{3}{1} = 3\) is HA

\(3x(x-1) = 0 \Rightarrow x = 0, x = 1\) are x-int

\(\frac{3(0)^2 - (0)}{(0)^2 + (0)-12} = 0 \Rightarrow y = 0\) is y-int
Example \[ \frac{3x^2 - 3x}{x^2 + x - 12} \]

We see the VA and HA

Example \[ \frac{2x^2 - 5x + 2}{x^2 - 4} \]

\[ \frac{2x^2 - 5x + 2}{x^2 - 4} = \frac{(2x-1)(x-2)}{(x+2)(x-2)} = \frac{2x-1}{x+2} \]

\((x+2)(x-2)=0 \Rightarrow x=-2, x=2 \) not in domain

\(x-2\) factors out, so \(x=2\) is hole
\(x+2\) does not factor out so \(x=-2\) is VA

\(n = d \Rightarrow y = \frac{2}{1} = 2\) is HA
\(2x-1 = 0 \Rightarrow x = \text{int } 1/2\)

\(2(0) - 1 = \frac{1}{2}\) is the y-int

**Asymptotes**

The x-axis is the horizontal asymptote when the degree of the numerator is less than the degree of the denominator

A horizontal asymptote other than the x-axis occurs when the numerator and the denominator have the same degree

The graph of a rational function may or may not cross a horizontal or oblique asymptote
Asymptotes

An asymptote is not part of the graph of the function.

We sketch asymptotes as dashed lines on graphs.